Direct Celestial Coordinates Solution for Cosmic Ray Shower Direction

Michael McEllin

1 Introduction

I want to take a different approach to solving for the primary angle of the cosmic ray shower, using 3D cartesian coordinates for each of the stations. In our XYZ coordinate system, the Z axis will be parallel to the Earth's rotation axis, the X axis will be the line from the Earth's centre to 0,0 latitude and longitude (i.e. on the equator and the Greenwich meridian). The Y direction with point East (perpendicular to the Z and X axes). These are called *Earth Centred Earth Fixed* coordinates. We choose this system because once we have worked out the direction of the shower, all we have to do to get celestial coordinates is a rotation around the Z axis.

We will assume that a number of stations, labelled by the index i, each register a cosmic ray shower event sufficiently closely spaced in time to be treated as a coincidence. Station i has x,y and z coordinates x_i, y_i, z_i. These are easy to work out from the geographical latitude,ϕ, longitude East, θ, and the Earth's radius, R. That is:

$$x = (R + \Delta R).cos(\phi).cos(\theta) \qquad (1)$$

$$y = (R + \Delta R).cos(\phi).sin(\theta) \qquad (2)$$

$$z = (R + \Delta R).sin(\phi) \qquad (3)$$

We have corrected for height above the ground by increasing the radius, R by the height ΔR. The event that is assumed to be part of a coincidence occurs at time t_i on detector i.

We now want to work out a shower direction in this coordinate system represented n the form of a vector $\overline{d}$ of unit length from the origin pointing to the arrival direction of the shower. (The unit length means that $\overline{d}.\overline{d} = d_x^2 + d_y^2 + d_z^2 = 1$, in terms of the vectors x,y,z components.)

In order for us to have a valid calculation we have to assume that the cosmic ray shower front moves at essentially the speed of light, which we will denote here by the conventional symbol used by physicists, c. Given the accuracy with which time differences can be measured this will be sufficiently good. The shower then moves as a narrow *front* - almost like a circular plate along the arrival direction. How do we express the mathematically?

2 Equation of Motion of a CR Shower Front

The general equation of a family of parallel 2D planes can be expressed as:

$$p.x + q.y + r.z = s \tag{4}$$

where the values of p, q and r (which define the orientation of the planes in space) stay the same and the value of the parameter s is allowed vary to distinguish the planes. We can insist that:

$$p^2 + q^2 + r^2 = 1 \tag{5}$$

so that p,q,r define a *unit* vector.

We can describe a cosmic ray shower front advancing through space (which constant orientation) by allowing the value of s to be a linear function of time. (Everything about the situation is linear.) We are therefore going to choose to write this as:

$$p.x + q.y + r.z = \alpha c.(t - t_o) \tag{6}$$

where c is the velocity of light and t_o is the unknown time at which the shower crosses the origin point of our coordinate system (since, clearly, we must have $t = t_o$ when $x = y = z = 0$).

Without loss of generality we can immediately simplify the presentation of our problem by redefining the origin of the coordinate system to that it rests at the first detector to be triggered and that we start measuring time from the time the first detector is triggered. (In fact, data from HiSPARC coincidences use exactly this convention.) We can immediately deduce that the value for α must be 1, because this would clearly have to be the case when the shower is moving directly along any of the axes.

I am also going to make a further simplification through scaling. Let us assume that we will measure distances in light-seconds (rather than meters etc.). Physicists call this a natural unit because it is related directly to a universal constant - the speed of light. When we measure distances like this the speed of light, c=1. (You could also just divide the whole equation through by c and then just work with variables x/c, y/c and z/c but our choice of units has a physically intuitive meaning.) This (along with other 'natural unit' adoptions) is a widespread convention in theoretical physics because it reduces symbol clutter in equations and allows the meaning of the equation to become more apparent.

Hence, using these natural units, we can just write:

$$p.x + q.y + r.z = t \tag{7}$$

providing we remember that we are now measuring distances in natural units and time starts when the shower touches the first detector, and the origin of position is also the first detector.

This equation has to hold whenever the shower front touches detector number i

(which has coordinates (x_i, y_i, z_i)) at time t_i. It will be convenient to start the counting from 0, that is the first detector to be triggered is detector 0. We now have a family of equations, one per detector:

$$p.x_i + q.y_i + r.z_i = t_i \tag{8}$$

but the equation for detector 0 is null - everything is 0, so it does not count.

With three detectors there are therefore two equations and three unknowns, except that we also have equation 5 which constrains the magnitude of the direction vector defined by (p,q,r) so we can still solve for the direction vector as I have shown elsewhere.

In the real World there are experimental errors, particularly in measuring arrival times at detectors, so we would generally be better to take account of as many detectors as are triggered by the shower. (If we rely on just three or four detectors the solution for the shower direction may well be very sensitive to small errors in the arrival times.) I am going to do this by writing and equation for the difference between the time the shower actually passed detector i (given by $p.x_i + q.y_i + r.z_i$) and the time actually logged t_i at detector i:

$$\epsilon_i = p.x_i + q.y_i + r.z_i - t_i \tag{9}$$

What we are going to do now is seek a solution for p, q and r that makes all the ϵ_i values for all the detectors as small as possible at the same time. With four detectors we could reduce to errors exactly to zero, but in general this is unlikely.

Most experimental measurement errors tend to be distributed 'normally', that is the probability of a particular size of error is described by a symmetrical bell-curve known as the 'normal' or 'Gaussian' probability distribution that is covered in A-level statistics. (There are some situations where you do not get normally distributed errors, but they are less common, and unless you have good evidence to the contrary a normal distribution is a good first assumption. Whether we are right in our assumption will become clear after we have obtained a solution - we just plot a histogram of the errors.) It can be shown (advanced maths) that for normally distributed errors the best way to estimate our unknown p, q and r values is to minimisation of the sum of squares of all the ϵ_i values. That is: minimise the sum S over n detectors of:

$$S = \Sigma_i^n (p.x_i + q.y_i + r.z_i - t_i)^2. \tag{10}$$

In order to take account of the $p^2 + q^2 + r^2 = 1$ constraint we need a trick called Lagrange multipliers. You will learn why this works in university maths. For now just accept that the magic works! It turns out we actually need to minimise the value of L, such that:

$$L = S - \lambda(p^2 + q^2 + r^2 - 1) \tag{11}$$

subject to variations in p,q,r *and* λ. λ is known as the Lagrange multiplier.

Squaring out the equation we get:

$$\begin{aligned} S = & \quad p^2.\Sigma_i^n x_i^2 + q^2.\Sigma_i^n y_i^2 + r^2.\Sigma_i^n z_i^2 + \Sigma_i^n t_i^2 \\ + & \quad 2p.q.\Sigma_i^n x_i.y_i + 2p.r.\Sigma_i^n x_i.z_i + 2q.r.\Sigma_i^n y_i.z_i \\ - & \quad 2p.\Sigma_i^n x_i.t_i - 2q.\Sigma_i^n y_i.t_i - 2r.\Sigma_i^n z_i.t_i \\ - & \quad \lambda(p^2 + q^2 + r^2 - 1) \end{aligned} \tag{12}$$

Note that all the sums are easy to work out computationally from the detector data, so this is just a quadratic in p, q, r .

We want to minimise this value by choice of appropriate values of p, q, r and λ. For this we need a bit of A-level - there no way around it. We need values of p, q, r that actually define a minimum so that any but the most infinitesimal change in p, q or r must *increase* the value of the sum S - that is what *minimum* means. Mathematically, we define this point by saying that the rate of change of S as we vary any of p, q, r, λ must be zero at the point of minimum S.

The rates of change of S with respect to changes in p, q, r and t_o are provided by a basic techniques of *calculus*, called differentiation. This is technically an easy step using straightforward mathematical rules, but since

I am not teaching calculus, I am just going to write down the formulas:

$$\begin{aligned}
\frac{\delta S}{\delta p} &= 2p.\Sigma_i^n x_i^2 + 2q.\Sigma_i^n y_i.x_i + 2r.\Sigma_i^n z_i.x_i - 2\Sigma_i^n x_i.t_i - 2\lambda p \\
\frac{\delta S}{\delta q} &= 2p.\Sigma_i^n x_i.y_i + 2q.\Sigma_i^n y_i^2 + 2r.\Sigma_i^n z_i.y_i - 2\Sigma_i^n y_i.t_i - 2\lambda q \\
\frac{\delta S}{\delta r} &= 2p.\Sigma_i^n x_i.z_i + 2q.\Sigma_i^n y_i.z_i + 2r.\Sigma_i^n z_i^2 - 2\Sigma_i^n z_i.t_i - 2\lambda r \\
\frac{\delta S}{\delta \lambda} &= p^2 + q^2 + r^2 - 1
\end{aligned} \tag{13}$$

We require that we must be at the point of minimum S by setting the $\delta S/\delta p$ etc. terms all to zero. (Note that when we do this the last equation is just our constraint $p^2 + q^2 + r^2 = 1$ - this is the point of introducing the Lagrange multiplier it automatically brings the constraint into play.)

When we have done this we might as well also divide everything through by 2 to get:

$$\begin{aligned}
p(\Sigma_i^n x_i^2 - \lambda) + q.\Sigma_i^n x_i.y_i + r.\Sigma_i^n x_i.z_i &= \Sigma_i^n x_i.t_i \\
p\Sigma_i^n x_i.y_i + q.(\Sigma_i^n y_i^2 - \lambda) + r.\Sigma_i^n y_i.z_i &= \Sigma_i^n y_i.t_i \\
p\Sigma_i^n x_i.z_i + q.\Sigma_i^n y_i.z_i + r.(\Sigma_i^n z_i^2 - \lambda) &= \Sigma_i^n z_i.t_i \\
p^2 + q^2 + r^2 &= 1
\end{aligned} \tag{14}$$

These are four equations in four unknowns. The system is solvable, but because it has products of variables ($p.\lambda$, $q.\lambda$, $r.\lambda$, and the quadratic form p^2 q^2 and r^2) it is non-linear, which makes it somewhat complicated. We managed this when we had just three detector stations in a coincidence, and we *could* also just about find it feasible to derive a 'direct' solution for the multi-station case using GCSE algebra but it would take a good deal of trouble to get the right answer and the formula would be very complex.

However, since we want numerical solutions anyway, we might as well use a numerical approximation procedure to get that solution. From a programming viewpoint it is less complicated and therefore more likely to be implemented correctly.

The basic idea here is to think of the value of L, defined by equation 11, as representing the height of the surface. (OK - difficult to visualise because it is four dimensional - but just think in two dimensions for the moment. The same idea follows through in four dimensions.) If we want to minimise L we start from some random point and look for a downhill direction. We know the downhill direction at any point, because the slope is given by the values when we substitution into equations 13. So, the procedure is to make a small steps in p, q, r and λ to reduce the value of L and we carry on doing this until we reach a point where any change starts to increase L again. This is sometimes known as the "method of steepest descents".

In practice, this is such a common task that there are well proven subroutines in standard numerical libraries which do the job very efficiently and robustly. There is no point in trying to program this yourself unless you are studying numerical methods and need the understanding. I used the *scipy.optimise.minimize* function from the exceedingly useful Python SCIPY library. (The NUMPY and SCIPY numerical methods libraries are the major reason scientists and engineers like working with Python.) We provide it with pointers to two subroutines, one evaluates equation 11, the other provides the four slope values from equation 13.

There is sometimes a problem with minimisation subroutines where the 'landscape' is a complicated shape and it is possible to get stuck in a 'local minima' that is not a low as the 'global optimum'. (Think of two valleys one lower than the other separated by a high ridge, Which valley you end up in will depend on which side of the ridge you start the steepest descent process.) Fortunately, in this case the L function is relatively well behaved. As it happens, most of the detectors in the Netherlands are on pretty much the same level plane, so it turns out that, like the 3-detector case, there are still two solutions, one pointing at the sky, one at the ground. However, we can easily avoid the ambiguity by starting our iterations at a point in the sky. We will alway find the sky solution this way rather than the ground solution.

At this point we should now have values for p, q and r.

3 Solving for Angles

Going back to the original equation of motion of the shower front:

$$p.x + q.y + r.z = t \tag{15}$$

we now need to think about the line perpendicular to the shower plane heading towards the origin. Where does this line touch the shower front?

Look at Figure 1 and note that the plane of the shower front must touch the x axis when y=0 and z=0, so x'=t/p. Similarly y'=t/q and z'=t/r. We can derive the coordinates of the point where the perpendicular touches the shower front, call this Γ, by noting that there are similar triangles $z'\Gamma O$ and $Z_\Gamma O\Gamma$, so:

$$\frac{z'}{t} = \frac{t}{tr} = \frac{t}{z_\Gamma} \implies z_\Gamma = r.t \tag{16}$$

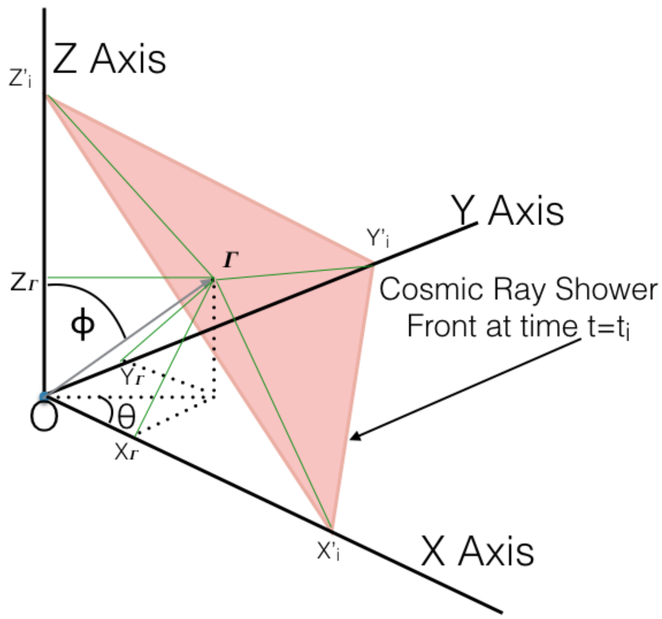

Figure 1: Cosmic Ray Shower Geometry

We could also derive in a similar way:

$$x_\Gamma = p.t, y_\Gamma = q.t. \tag{17}$$

Hence, the angle with the z axis, ϕ, is given by:

$$cos(\phi) = \frac{z_\Gamma}{z'} = r.t/t \implies cos(\phi) = r \tag{18}$$

It follows from trigonometric identities that $p^2 + q^2 + r^2 = 1$ (which in fact was always immediately obvious from the earlier definitions of the equation of the wave front and A-level vector analysis - but you have not done that yet).

The angle with the x axis, θ is determined by the ratio of the y and x coordinates of our point T, so:

$$tan(\theta) = q/p \tag{19}$$

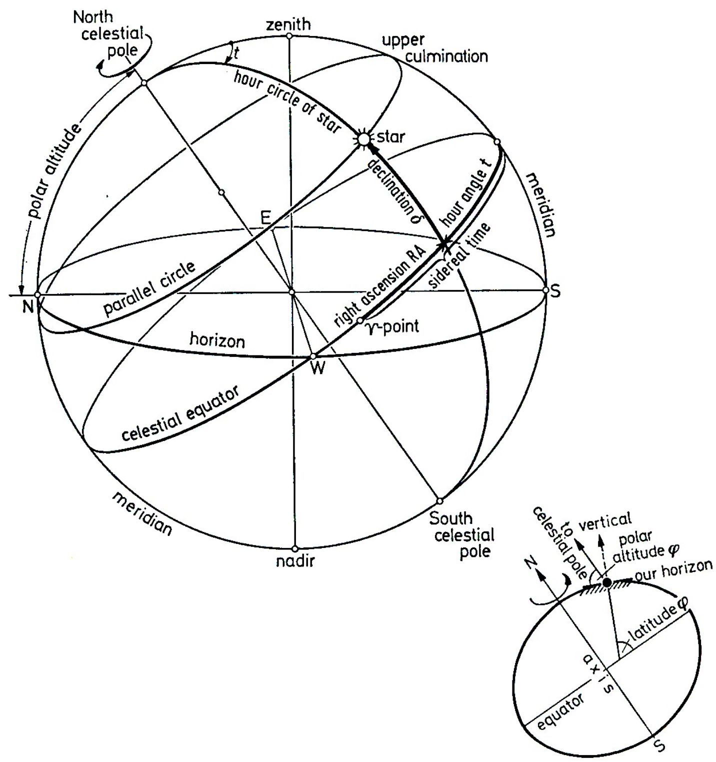

Figure 2: Celestial Coordinates.

4 Solving for Right Ascension and Declination

Our angles θ and ϕ are defined with respect to the ECEM coordinate system, which rotates with the Earth. That is useful for some purposes, but if we wish to relate show directions to fixed directions on the sky, then we need to convert from this rotating system to the fixed Celestial Coordinate system. (See figure 2 on page 8 taken from Unsöld (Unsöld 1969).)

Note that since the z axis by definition has been aligned with the Earth's axis, θ is easily related to the astronomical declination. (The declination, conventionally denoted by the symbol δ, is defined as the angle with the equatorial plane: so $\delta = \pi/2 - \theta$, or alternatively $sin(\delta) = r$.

The angle θ is of course *not* the astronomical *Right Ascension* coordinate (usually denoted by the symbol α) because our xyz coordinate system is rotating with the Earth, and the projected x-axis sweeps round the Earth's equatorial plane projected onto the sky once every sidereal day.

The next bit of reasoning is basically straightforward - but it is easy to get confused and use the wrong signs so it helpful to write down a number of precise definitions.

the *North Celestial Pole* is the projection of the Earth's rotation axis through its North Pole onto the sky. Observationally, in the northern hemisphere, all the starts appear to rotate around this point (which is approximately marked by Polaris, the pole star).

Zenith is the point directly over your head.

Meridian is the great circle on the sky running from the north celestial pole, though the meridian, down to due South. If you stand looking due South and move your head up and down, your point of view is moving up and down the meridian.

Hour Circle of an astronomical object is the great circle on the sky running through the north celestial pole down through the position of the object of interest.

Hour Angle (HA) is the angle between the meridian and the hour circle of the object. It can be measured in degrees, or hours, minutes and seconds. (The use of time as a measurement is very convenient when doing astronomical observations because the hour angle is just the amount of time that has passed since the object rotated through the meridian. Since astronomical object appear to move from East to West on the sky, hour angle increases in the westerly direction.

Vernal Equinox also known as the *First Point of Aries*, denoted by the old astrological symbol ♈, is the point on the Celestial Equator from which *Right Ascension* coordinates are measured. (It is actually defined as the point where the Sun crosses the Celestial Equator in the Spring.) Right ascension is conventionally measured in hours minutes and seconds (and this is always used in celestial catalogues) but is easily converted to degrees, since $360^o = 24$ hours. Right ascension increases going round to the east (or looking down from the north pole it increases going anticlockwise). The *First Point of Aries* crosses the meridian when the local sidereal time (LST) is 0 hours. If we wait until the sidereal time on our astronomical clock equals the right ascension of our object the we will find the object right on the meridian.

Sidereal Time This is a time standard that allows astronomers to find celestial objects easily. If you observe a star at the same 'wall clock' or 'Solar' time night after night then you would see that its apparent position (that is, relative to your local NSEW and vertical directions) on the sky changes progressively. (In fact, it would go all around the sky taking a year to come back to the same positon). Sidereal Time is defined so that at the same Sidereal Time and object has the same position on the sky. The Sidereal Day is slightly shorter than the Solar Day (23 hours 56 minutes 4.0916 seconds). That is, when our Sidereal clock reaches 23h54m4.09s we start counting a new day from 0h0m0s. Astronomical observatories run their observing programs by Sidereal Time. (When I was a research student doing observing duty on the Cambridge One Mile Radio Telescope, I had to do all my setting-up work four minutes earlier each day. Fortunately, we used to plan our observing programs so that the objects that we were following were visible in the sky during a normal working day, so I never had to work nights, like the optical astronomers.) We need to distinguish two types of Sidereal Time:

- ***Local Sidereal Time* (LST)** This is defined such that the *First Point of Aries* crosses the meridian at 0 hours. LST is important because if an astronomical object has a Right Ascension (see below) of 1hour, it will cross the meridian one hour after the start of the Sidereal Day at 0 hours.
- ***Greenwich Mean Sidereal Time* (GMST)** This is defined such that the *First Point of Aries* (a fixed reference point on the sky - see below) crosses the meridian at Greenwich at 0 hours.

Using the above definitions it is possible to deduce the following simple relationship:

$$Right\ Ascension = LST - HourAngle \tag{20}$$

We have actually defined our xyz coordinate system as if we were standing on the Greenwich Meridian (the line of 0 longitude) since we required our x axis to run from the centre of the Earth through the point of 0 longitude, 0 latitude on the equation. Even though many of our detectors are not actually on this meridian, everything refers back to it. Hence, our local sidereal time is the same as Greenwich Mean Sidereal Time At this point it is easiest to look up standard formula for Greenwich Mean Sidereal Time, as used by astronomers.

$$GMST = (18.697374558 + 24.06570982441908D)modulus(24) \tag{21}$$

where D is the number of days (including fractions of a day) since 12 Noon (Universal Time) on 1st January in the year 2000. This is known as the *Julian Day* with respect to the year 2000 epoch[1].

Using Julian Days to log astronomical observations avoids difficulties with working out time intervals over periods including leap years and so on. The day starts at noon because until recently most astronomy took place at night and it was inconvenient to have a single series of observations on the same night being logged as if they occured on two different days. This reasoning is now less compelling than it was, since radio telescopes and satellite observatories work 24 hours a day, and most observations are now made with the help of computers anyway, but we are stuck with the convention.

The modulus operation reduces the GMST value to the range 0...24. (That is, we take out any whole multiples of 24 from the answer.) Various sources on the Web, including the Wikipedia Julian Day page, provide algorithms for determining Julian Day from the calendar date, but for single observations it often easiest just to look up one of the on-line conversion utilities such as, for example, from the US Naval Observatory.

Hence if we know the the Julian Day and the hour angle we can find the right ascension. We have calculated a value of θ, which is the angle going eastwards from the x axis to the hour circle of our shower direction. The hour angle is measured positive going westwards, so the hour angle is just $-\theta$.

References

Unsöld, A. (1969), *The New Cosmos*, Springer.

[1]Beware! If you come across the term Julian Day be careful that you get the right *epoch*, that is the reference point from which time is measured. There are at least nine difference epoch definitions used at various times by various subgroups of the astronomy community. The original definition of Julian Days used noon on January 1st 4713BC since it is the starting point of a number of astronomical cycles and also before any historical date associated with an astronomical observation.

The trouble with this definition is that the numbers become large. For example the Julian Day number for 12:00 UT on January 1, 2000, was 2,451,545. Hence, it is often more convenient when tabulating data to refer Julian Days to a later epoch (such as the start of the new millennium). As explained earlier there are many possible choice found in the astronomical literature. You need to check.