Direct Celestial Coordinates Solution for Cosmic Ray Shower Direction - Alternative Solution Schemes

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1 Introduction

We will use the 3D cartesian coordinates for each of the stations in order to do a direct solution for cosmic ray shower direction in a coordinate system that is easy to relate to celestial coordinates. In our XYZ coordinate system, the Z axis will be parallel to the Earth's rotation axis, the X axis will be the line from the Earth's centre to 0,0 latitude and longitude (i.e. on the equator and the Greenwich meridian). The Y direction with point East (perpendicular to the Z and X axes). These are called *Earth Centred Earth Fixed* coordinates. We choose this system because once we have worked out the direction of the shower, all we have to do to get celestial coordinates is a rotation around the Z axis.

The previous direct solution scheme that I described has two disadvantages: firstly, the algebra is moderately complex; secondly, it generates a solution for a ratio Q/P, where Q is the projection of the a unit vector describing the shower direction onto the Y axis, and P is the projection of this vector onto the X axis. The problem here is that when we wish to calculate the angles describing the shower direction (hour angle in the equatorial plane and declination the angle made with this plane) there is an ambiguity in that hour-angle = arctan(Q/P), which will produce the same result if Q and P are both inverted in sign. One needs to disambiguate solutions by checking which of the two ambiguous directions generates the known time delays which specify the problem.

The scheme described here is in essence very similar, but it keeps the unknowns P, Q and R separate. This fortunately leads to algebra which is a bit (but not much) simpler, but comes with a different complication. As before, we have to eliminate two of P, Q and R to solve for the remaining value before back-substitution for the other values. We can choose to solve for any of P, Q or R, but for each choice (and its associated elimination scheme) there are detector geometries lead to undefined solutions. This means that we need to be prepared to switch the choice of variable elimination according to the detector configuration with which we are dealing. This is not much of a problem, but it does mean that we have to write down three alternative solution schemes.

The three stations A, B, C are assumed to have coordinates (a_x, a_y, a_z), (b_x, b_y, b_z) and (c_x, c_y, c_z). These are easy to work out from the geographical latitude,ϕ, longitude East, θ, and the Earth's radius, R. That is:

$$x = (R + \Delta R).cos(\phi).cos(\theta) \quad (1)$$
$$y = (R + \Delta R).cos(\phi).sin(\theta) \quad (2)$$
$$z = (R + \Delta R).sin(\phi) \quad (3)$$

We have corrected for height above the ground by increasing the radius, R by the height ΔR.

We now want to work out a shower direction in this coordinate system represented in the form of a vector $\overline{d}$ of unit length along the arrival direction of the shower.

Let us assume that the shower is first detected at station A, at time t=0, and subsequently at station B at time t_B and station C at time t_C. (That is, positive values of t_B or t_C are associated with time *delays* at detectors B and C. Negative values of t_B or t_C mean that the shower is seen first at detector B or detector C. See Figure 1 for the geometry, Note that the shower in this diagram is moving bottom left to top right along the line of the grey arrow.

From now on, in fact, we are going to assume that distances are measured in light-seconds. This simplifies the subsequent algebra and normalises the computer calculations by removing a velocity of light term. It also increase the robustness of the numerical manipulations because for most situations of interest the time delays t_B and t_C will be comparable in magnitude to the inter-detector distances, when they are measured in light seconds. This makes it possible to avoid small differences between large values, or ratios of quantities of very different magnitudes, both of which can produce numerical inaccuracies through rounding errors.

In order for us to have a valid calculation we have to assume that the cosmic ray shower front moves at essentially the speed of light, which we will denote here by the conventional symbol used by physicists, c. Given the accuracy with which time differences can be measured this will be sufficiently good. The shower then moves as a narrow *front* - almost like a circular plate

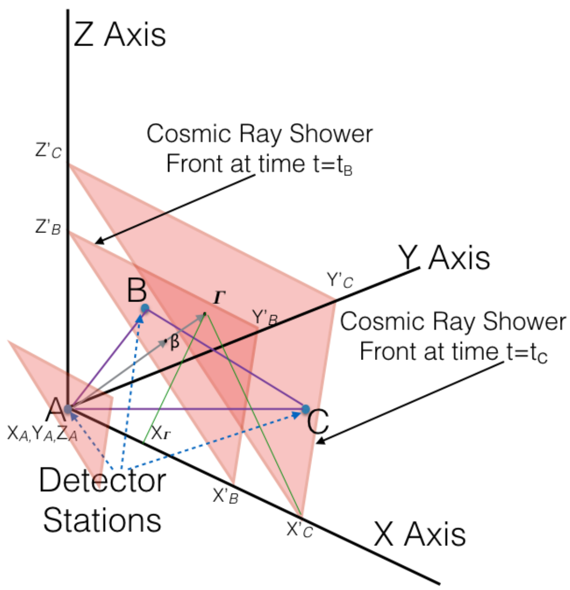

Figure 1: Cosmic Ray Shower Geometry

touching first A, then B then C. So, the distance AB' in the diagram will be $c.t_B$ and AC' will be $c.t_C$. How do we express these distances in terms of the angle at which the shower is moving?

2 Equation of Motion of a CR Shower Front

The general equation of a family of parallel 2D planes can be expressed as:

$$P.x + Q.y + R.z = s \tag{4}$$

where the values of P,Q and R stay the same and the value of s changes to distinguish the planes. Note that there is a certain amount of ambiguity in these values: we could divide through the equation by a constant and still get effectively the same set of planes.

In fact, since x, y and z are measured in light seconds, we can immediately choose to require that $P^2 + Q^2 + R^2 = 1$, that is we have define a unit vector whose individual components can only vary between the limits -1 and +1, while its total length remains constant It is obvious that s must then be just the time delay from the origin or x, y and z (i.e detector A). (To see this, note that $Q = R = 0 \implies P = 1, x = s$ i.e. the time delay for a signal travelling from detector A along the x axis to the detector a distance x light-seconds along. Similar conclusions hold for the y and z axes.)

We must also remember that the direction of this unit vector is along the direction of movement of the shower (that it, it points *from* a location in the sky, not *to* a location in the sky). This point is important when converting from hour angles (ha) and declinations (δ) to P,Q,R denoted directions. We also need to remember that hour-angle is measured as positive westward, whereas our right-handed coordinate system has the Y direction pointing East. Hence, the relationship between a point of the sky which is the origin of a shower and P,Q and R must be:

$$P = -cos(\delta).cos(ha) \tag{5}$$
$$Q = -cos(\delta).sin(ha) \tag{6}$$
$$R = -sin(\delta). \tag{7}$$

The inverse relationship is:

$$ha = atan(Q/P) \tag{8}$$
$$\delta = asin(-R) \tag{9}$$

Note that since declination varies from -90^o to $+90^o$ only there is no ambiguity in the conversion from R to a declination. This is not necessarily the case for the inverse tangent where there is an ambiguity of π. It is necessary to take account of the individual signs of P and Q to obtain the right hour-angle value. (However, when programming the solution, use of the `atan2(Q,P)` library routine usually takes account of the signs to get the right soluiton.)

Hence, subsituting the positions of detector B and detector C into the equation of the shower front we get:

$$t_B = P.X_B + Q.Y_B + R.Z_B \tag{10}$$
$$t_C = P.X_C + Q.Y_C + R.Z_C \tag{11}$$

with:

$$P^2 + Q^2 + R^2 = 1 \tag{12}$$

Note that this is rather simpler reasoning that that used to derive equations similar to 10 and 11 in the earlier document. Here, I am just assuming what is in truth a rather obvious linear relationship, which is an immediate consequence of A level vector analysis principles (rather than deriving some of those results in long-winded way).

We have three equations and three unknowns, so a solution is possible.

We can then solve for angles using:

$$R = sin(declination) \tag{13}$$

$$Q/P = -tan(hour\ angle) \tag{14}$$

The minus sign for the solution for hour-angle arises because hour-angles are measured westwards, while our Y axis is defined positive eastwards. Note the potential ambiguity in evaluation of the inverse tangent is resolved by reference to the individual signs of P and Q. (In programmatic terms, we use the *atan2(Q,P)* function.)

3 Solutions

We can eliminate Q from equations 10 and 11 to get:

$$P.(X_B Y_C - X_C Y_B) + R.(Z_B Y_C - Z_C Y_B) = (t_B Y_C - t_C Y_B) \tag{15}$$

We can eliminate P from equations 10 and 11 to get:

$$Q.(Y_B X_C - X_B Y_C) + R.(Z_B X_C - Z_C X_B) = (t_B X_C - t_C X_B) \tag{16}$$

Eliminating R would give:

$$P.(X_B Z_C - X_C Z_B) + Q.(Y_B Z_C - Y_C Z_B) = (t_B Z_C - t_C Z_B) \tag{17}$$

Remember that these equations are not independent: any two of the three represent the same information as equations 10 and 11. We do, however, need the three equations because it is possible for the coefficient terms to become zero for particular detector configurations. For example, if all the detectors are in the X/Y plane with $Z_B = Z_C = 0$, then equation 17 becomes $0 = 0$ and this is no further use. (However, as long as the three detectors are not in a line the other two equations must then have non-zero coefficients.)

The possibility of any one of the equations disappearing for particular configurations means that in general, for any particular set of data, we need to choose between three possible ways of moving forwards.

There are more advanced methods in matrix algebra that could deal with these issues without explicitly having to write down and choose between three solution schemes. Here, however, I have chosen to make technique explicit using elementary algebra.

3.1 Scheme 'R'

This scheme is valid whenever: $(Y_B X_C - X_B Y_C) \neq 0$. We can reformulate equation 15

$$P.(X_B Y_C - X_C Y_B) + R.(Z_B Y_C - Z_C Y_B) = (t_B Y_C - t_C Y_B)$$

as:

$$P = -R.\frac{(Z_B Y_C - Z_C Y_B)}{(X_B Y_C - X_C Y_B)} + \frac{(t_B Y_C - t_C Y_B)}{(X_B Y_C - X_C Y_B)}. \tag{18}$$

For the sake of simplicity of substitution later on, I will define two coefficients consisting entirely of known quantities:

$$U_{PR} = -\frac{(Z_B Y_C - Z_C Y_B)}{(X_B Y_C - X_C Y_B)} \tag{19}$$

$$V_{PR} = +\frac{(t_B Y_C - t_C Y_B)}{(X_B Y_C - X_C Y_B)}. \tag{20}$$

or:

$$P = R.U_{PR} + V_{PR} \tag{21}$$

Equation 16, which is:

$$Q.(Y_B X_C - X_B Y_C) + R.(Z_B X_C - Z_C X_B) = (t_B X_C - t_C X_B)$$

becomes:

$$Q = -R.\frac{(Z_B X_C - Z_C X_B)}{(Y_B X_C - X_B Y_C)} + \frac{(t_B X_C - t_C X_B)}{(Y_B X_C - X_B Y_C)}. \tag{22}$$

Again I define two coefficients consisting entirely of known quantities:

$$U_{QR} = -\frac{(Z_B X_C - Z_C X_B)}{(Y_B X_C - X_B Y_C)} \tag{23}$$

$$V_{QR} = +\frac{(t_B X_C - t_C X_B)}{(Y_B X_C - X_B Y_C)}. \tag{24}$$

So

$$Q = R.U_{QR} + V_{QR}. \tag{25}$$

Since both $P = R.U_{QR} + V_{PR}$ and $Q = P.U_{QR} + V_{QR}$ are well-defined we substitute for P and Q into equation 12 ($P^2 + Q^2 + R^2 = 1$) and get:

$$(U_{PR}.R + V_{PR})^2 + (U_{QR}.R + V_{QR})^2 + R^2 = 1 \tag{26}$$

Which expands to:

$$(U_{PR}^2+U_{QR}^2+1)R^2+2(U_{PR}.V_{PR}+U_{QR}.V_{QR}).R+V_{PR}^2+V_{QR}^2-1=0 \quad (27)$$

The normal method of solving quadratics then gives

$$R=\frac{-((U_{PR}.V_{PR}+U_{QR}.V_{QR})}{(U_{PR}^2+U_{QR}^2+1)}\pm\frac{1}{2}\sqrt{\frac{(U_{PR}.V_{PR}+U_{QR}.V_{QR})^2}{(U_{PR}^2+U_{QR}^2+1)^2}+4\frac{(1-V_{PR}^2-V_{QR}^2)}{(U_{PR}^2+U_{QR}^2+1)}} \quad (28)$$

We can then back-substitute for P and Q using equations 21 and 25.

Note that the discriminant (the expression inside the square root sign) will be positive for all values of time delays that are consistent with the same shower front progressing across the detectors at the speed of light. It would become negative if, for example, the triggering detectors B and/or C took place later than the longest possible light-transit time in the detector geometry. A negative discriminant value is therefore always an indication that we do not have a true coincidence detection.

3.1.1 Sanity Check

We can do a sanity check on this equation. Let us assume that all the detectors are in the XY plane, Station B is on the X axis and Station C on the Y axis (Station A is effectively the origin). This means that

$$Z_A = Z_B = Z_C = 0 \quad (29)$$

$$Y_B = 0 \quad (30)$$

$$X_C = 0 \quad (31)$$

From the equations above we find that:

$$Q=\frac{t_C}{Y_C}; P=\frac{t_B}{X_B} \quad (32)$$

- $t_B=0, t_C=0 \implies Q=0, P=0 \implies R=1$.
- $t_B=0, t_C\neq 0 \implies Q=t_C/Y_C, P=0$.
- $t_C=0, t_B\neq 0 \implies P=t_B/X_B, Q=0$.
- $t_B=t_C \implies P/Q=X_B/Y_C$, and if $X_B=Y_C$ then $P=Q$.
- $t_C=Y_C, t_B=0 \implies Q=1, P=0, R=0$.

- $t_B = X_B, t_C = 0 \implies Q = 0, P = 1, R = 0.$

All this looks sensible when interpreted as solutions.

We can also assume another special geometry, with $Z_C = Z_B = 0$. This immediately implies that $U_{PR} = U_{QR} = 0$ so that the quadratic reduces to

$$R = \pm\sqrt{1 - V_{PR}^2 - V_{QR}^2} \tag{33}$$

If $t_B = t_C = 0$ (that is a shower approaching down the Z axis perpendicular to the plane of the detectors) we get $R = \pm 1$. This is exactly as we would expect, since it also necessarily implies P = Q = 0.

For $t_B = 0, t_C \neq 0, X_B, Y_C \neq = 0, X_C = Y_B = 0$ we get:

$$Q = V_{QR} = \frac{t_C X_B}{Y_C}, P = 0, R = \pm\sqrt{1 - Q^2} \tag{34}$$

Again, this all makes complete sense.

3.2 Scheme 'Q'

This scheme takes account of detectors entirely in the X-Z plane (i.e. $Y_B = Y_C = 0$). It is valid whenever: $(X_B Z_C - X_C Z_B) \neq 0$. We will aim to end up with a quadratic equation for Q so we need to find equations for P and R in terms of Q.

We can rewrite equation 17:

$$P.\left(X_B Z_C - X_C Z_B\right) + Q.\left(Y_B Z_C - Y_C Z_B\right) = \left(t_B Z_C - t_C Z_B\right)$$

as:

$$P = -Q.\frac{(Y_B Z_C - Y_C Z_B)}{(X_B Z_C - X_C Z_B)} + \frac{(t_B Z_C - t_C Z_B)}{(X_B Z_C - X_C Z_B)}. \tag{35}$$

Again we can write:

$$U_{PQ} = -\frac{(Y_B Z_C - Y_C Z_B)}{(X_B Z_C - X_C Z_B)} \tag{36}$$

$$V_{PQ} = +\frac{(t_B Z_C - t_C Z_B)}{(X_B Z_C - X_C Z_B)}. \tag{37}$$

So, we can now write:

$$P = Q.U_{PQ} + V_{PQ} \tag{38}$$

Similarly, from equation 16, which is:

$$Q.\,(Y_BX_C - X_BY_C) + R.\,(Z_BX_C - Z_CX_B) = (t_BX_C - t_CX_B)$$

$$R. = -Q.\frac{(Y_BX_C - X_BY_C)}{(Z_BX_C - Z_CX_B)} + \frac{(t_BX_C - t_CX_B)}{(Z_BX_C - Z_CX_B)} \tag{39}$$

Again we can write:

$$U_{RQ} = -\frac{(Y_BX_C - X_BY_C)}{(Z_BX_C - Z_CX_B)} \tag{40}$$

$$V_{RQ} = +\frac{(t_BX_C - t_CX_B)}{(Z_BX_C - Z_CX_B)} \tag{41}$$

and

$$R = Q.U_{RQ} + V_{RQ} \tag{42}$$

Substituting for P and R into equation 12 we get

$$(U_{PQ}.Q + V_{PQ})^2 + (U_{RQ}.Q + V_{RQ})^2 + Q^2 = 1 \tag{43}$$

Which expands to:

$$(U_{PQ}^2 + U_{RQ}^2 + 1)Q^2 + 2(U_{PQ}.V_{PQ} + U_{RQ}.V_{RQ}).Q + V_{PQ}^2 + V_{RQ}^2 - 1 = 0 \tag{44}$$

The normal method of solving quadratics then gives

$$Q = \frac{-((U_{PQ}.V_{PQ} + U_{RQ}.V_{RQ})}{(U_{PQ}^2 + U_{RQ}^2 + 1)} \pm \frac{1}{2}\sqrt{\frac{(U_{PQ}.V_{PQ} + U_{RQ}.V_{RQ})^2}{(U_{PQ}^2 + U_{RQ}^2 + 1)^2} + 4\frac{(1 - V_{PQ}^2 - V_{RQ}^2)}{(U_{PQ}^2 + U_{RQ}^2 + 1)}} \tag{45}$$

We can then back-substitute for P and R using equations 38 and 42.

3.3 Scheme 'P'

This scheme is valid whenever: $(Y_BZ_C - Y_CZ_B) \neq 0$ and takes account of the possibility of detectors entirely in the YZ plane (i.e. $X_B = X_C = 0$). We will aim to end up with a quadratic equation for P so we need to find equations for Q in terms of P and R in terms of P.

Equation 17, which is:

$$P.\,(X_BZ_C - X_CZ_B) + Q.\,(Y_BZ_C - Y_CZ_B) = (t_BZ_C - t_CZ_B)$$

can be rewritten as:

$$Q = -P.\frac{(X_B Z_C - X_C Z_B)}{(Y_B Z_C - Y_C Z_B)} + \frac{(t_B Z_C - t_C Z_B)}{(Y_B Z_C - Y_C Z_B)}. \tag{46}$$

Again we can write:

$$U_{QP} = -\frac{(X_B Z_C - X_C Z_B)}{(Y_B Z_C - Y_C Z_B)} \tag{47}$$

$$V_{QP} = +\frac{(t_B Z_C - t_C Z_B)}{(Y_B Z_C - Y_C Z_B)}. \tag{48}$$

So, we can now write:

$$Q = P.U_{QP} + V_{QP} \tag{49}$$

Similarly, from equation 15, which is:

$$P.\,(X_B Y_C - X_C Y_B) + R.\,(Z_B Y_C - Z_C Y_B) = (t_B Y_C - t_C Y_B)$$

we can immediately write:

$$R = -P.\frac{(X_B Y_C - X_C Y_B)}{(Z_B Y_C - Z_C Y_B)} + \frac{(t_B Y_C - t_C Y_B)}{(Z_B Y_C - Z_C Y_B)} \tag{50}$$

Again we can write:

$$U_{RP} = -\frac{(X_B Y_C - X_C Y_B)}{(Z_B Y_C - Z_C Y_B)} \tag{51}$$

$$V_{RP} = +\frac{(t_B Y_C - t_C Y_B)}{(Z_B Y_C - Z_C Y_B)} \tag{52}$$

and

$$R = P.U_{RP} + V_{RP} \tag{53}$$

Substituting for P and R into equation 12 we get

$$(U_{RP}.P + V_{QP})^2 + (U_{RP}.P + V_{RP})^2 + P^2 = 1 \tag{54}$$

Which expands to:

$$(U_{RP}^2 + U_{RP}^2 + 1)P^2 + 2(U_{RP}.V_{RP} + U_{QP}.V_{QP}).P + V_{RP}^2 + V_{QP}^2 - 1 = 0 \tag{55}$$

The normal method of solving quadratics then gives

$$P = \frac{-((U_{QP}.V_{QP} + U_{RP}.V_{RP})}{(U_{QP}^2 + U_{RP}^2 + 1)} \pm \frac{1}{2}\sqrt{\frac{(U_{QP}.V_{QP} + U_{RP}.V_{RP})^2}{(U_{QP}^2 + U_{RP}^2 + 1)^2} + 4\frac{(1 - V_{QP}^2 - V_{RP}^2)}{(U_{QP}^2 + U_{RP}^2 + 1)}} \tag{56}$$

We can then back-substitute for Q and R using equations 49 and 53.

3.4 Choice of Solution Scheme

This is a relatively easy choice. The conditions for valid solutions for each scheme are:

Scheme 'R' : $(Y_B X_C - X_B Y_C) \neq 0$

Scheme 'Q' : $(X_B Z_C - X_C Z_B) \neq 0$

Scheme 'P' : $(Y_B Z_C - Y_C Z_B) \neq 0$

These are terms that appear in fraction denominators. Hence we simply choose the scheme where the conditional term has the largest absolute value.

In practice, for real HiSPARC detector clusters, situated at mid-latitudes, more or less in the local horizontal plane (that is vertical coordinates small compared with horizontal inter-station distances) it is likely that all of the solution schemes would produce valid results.

It is nevertheless worth ensuring that the solution methodology can cope with all possible configurations because it is useful in testing to present test date that has solutions which can be straightforwardly checked manually. These are most likely to be configuration where one or more of the optional solution schemes tend to break down.