Parallax Baseline for Moon Distance Measurement

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Introduction

We need to establish a distance between two points on the Earth that will act as a baseline for a parallax measurement. We are trying to determine the small acute angle subtended at the Moon by the two measuring locations. Ideally, we would like the two measurement points to make an isosceles triangle with the centre of the Moon to make the maths simple. Unfortunately, this would only be true for special pairs of points on the Earth's surface. In general, one of the points will be closer to the Moon than the other, and the triangle formed by the Moon and the two measurement points will have an irregular shape.

It is therefore clearly not enough just to calculate the absolute distance between the two ends of the baseline. What we need to do is to think of the line joining the Earth and Moon, and then imagine a plane perpendicular to this line, passing through the centre of the Earth. We now have to extend the lines from the Moon to our observation points forwards until they intersect our imagined plane, giving two "projected" points. It is the distance between these projected points that will determined our parallax baseline, allowing us to calculate the distance from the centre of the Earth to the centre of the Moon.

This is not a trivial calculation, but it is far from impossible. The easiest way to do this is in fact to do a bit more work than we strictly need to do, going to an x,y,z coordinate system and then thinking of the problem in terms of rotating the coordinate system to make the problem look easy - putting one axis along the line from the Earth to the Moon. This trick of shifting a coordinate system to make a problem look easy turns up frequently whenever computers are appled to science/engineering problems, so all the methods based on matrix algebra have been fully worked out and codified - first year university stuff for maths and physics undergraduates. We are less likely to stray from the path if we stick to well-used calculation routes, even if it is slightly further to get round to where we want to be.

Calculation

Part 1: Converting Lattitude and Longitude to X,Y,Z Coordinates

Trigonometry in 3D is tough on the brain. It will help if we first convert all the locations with which we are dealing into a cartesian (x,y,z) coordinate system because we can then use Pythagoras. So, we will assume we are dealing in fractional degrees (e.g. as obtained from Google Earth or one of the other on-line map systems[1]), with lattitude denoted by the variableθ conventionally positive from the equator north, and ϕ measuring longitude positive going eastwards from the Greenwich meridian (the part of the great circle at 0 longitude running from pole to pole through the equator). We will make our x,y plane coincide with the equator and the x,z plane run through the poles and the Greenwich meridian (and the International Date Line). I will distinguish all the coordinates associated with SHS by a subscript "n" (for North) and those for the South African school with "s" (for South). Hence, if R_E is the radius of the Earth:

$$x_n = R_E cos\left(\theta_n\right) cos\left(\phi_n\right)$$

$$y_n = R_E cos(\theta_n) sin(\phi_n)$$

$$z_n = R_E sin(\theta_n)$$

$$x_s = R_E cos\left(\theta_s\right) cos\left(\phi_s\right)$$

$$y_s = R_E cos(\theta_s) sin(\phi_s)$$

$$z_s = R_E sin(\theta_s)$$

Clearly, then, the absolute distance between these two points is just an application of Pythagoras:

$$D = \sqrt{(x_n - x_s)^2 + (y_n - y_s)^2 + (z_n - z_s)^2}$$

However, though this is interesting to know, it is not what we need, because in general the direct line between the two schools is not likely to be perpendicular to the line between the centre of the Earth and the centre of the Moon. The real parallax baseline with be foreshortened by a certain amount that will depend on a complicated way on the geometry of the Earth, the two schools and the Moon. (The two schools are not equally distances above and below the equator, and the Moon itself normally sits either above or below the plane of the equator. It will also be a bit to the east or west of the observing stations a the time the photographs are taken.) This is all quite difficult to visualise, and it is also extremely tricky to produce a clear illustration, because it really needs to be in stereoscopic 3D.

[1] The coordinates of Stroud High School are lat=51.747, long=-2.230 to sufficient accuracy.

Part 2: Projection to Get the Parallax Baseline

What we need to do here is to rotate our original x,y,z coordinate system to a new x',y',z' system, where the x-axis lies along the line between the Earth and the Moon. We can choose to keep our y' axis in the plane of the equator, for simplicity, since all we require is the y' and z' are perpendicular to the Earth/Moon vector.

So, where is the Earth/Moon vector, in our original x,y,z coordinate system? For this we will need some help from Stellarium. We use the Stellarium *Location* window (move the mouse down to the lower left of the Stellarium window, then up to the top of the toolbox that appears). Use the pop-up window to the set the sky view to be that on the equator and the Greewich meridian (0.0^o lattitude , 0.0^o longitude). Also use the *Date/Time* window to set the time at which the observation are to be (or have been) made. Now find the Moon, and click on it. You will then see a data table at the top right of the screen, in which one line will begin with "***Az/Alt***"[2]. This stands for Azimuth and Altitude, and the two numbers show the position of the Moon in the sky as seen from this point. Azimuth is measured going round from North at 0^0, East at 90^o,South at 180^o and West at 270^o. Altitude is the height in the sky above the horizon, again in degrees. Given that we picked a special point on the equator, you should be able to see that the azimuth is just the longitude at which the Earth/Moon vector would pass through the Earth's surface. Call this point "m". We will call the azimuith/longitude ϕ_m. The altitude, θ_m, is also related to the latitude of the same point - but careful here! If the azimuth is between 0^0 and 180^othen $lattude = altitude$, but if azimuth is between $180^o$and 360^othen $lattitude = -altitude$.

What we need to do now is to rotate our original coordinate system so that the x-axis lies on the equator along the line of longitude that runs through point m. Obviously this angle of rotation is just the value of the longitude of point m. We then rotate around the y-axis (which stays on the equator) bringing the x-axis up so that it lies through point m. This rotation around the new y axis is clearly just the lattitude of point m. At this point the new z-axis is leaning back from its original orientation along the Earth's rotation axis. Once we have done all this and read off the new x',y',z' coordinates of the two schools, we just ignore the x' coordinates and use Pythagoras on the new y',z' coordinates to get the projected baseline for a parallax measurement.

There is no use denying that coordinate rotation is a bit tricky, but it is a very standard mathematical procedure for anyone studying maths or theoretical physics at university. We could look up the answer we need, but it is fairly easy to see what it must be in our somewhat simplified problem (remember we keep y' in the original x,y plane).

The new coordinates will always be expressible in terms of the old coordinates by a 3x3 matrix equation (where we have to work out the r_{ij}):

[2] If you do not see Az/Alt, then you need to go into the Configuration options, where you will find a checkbox that requests that this particular data should be displayed in the table.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

However, the easiest way to get to this is to think about a rotation in 2D first. It is fairly easy to show by elementary trig that a rotation by in the x,y plane,ϕ_m, can be represented by:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} cos(\phi_m) & sin(\phi_m) & 0 \\ -sin(\phi_m) & cos(\phi_m) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Similarly, a rotation by θ_m around a y-axis (in the x,z plane must by analogy be:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} cos(\theta_m) & 0 & sin(\theta_m) \\ 0 & 1 & 0 \\ sin(\theta_m) & 0 & cos(\theta_m) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Applying first theϕ_m and then θ_m rotation is then just:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} cos(\theta_m) & 0 & sin(\theta_m) \\ 0 & 1 & 0 \\ sin(\theta_m) & 0 & cos(\theta_m) \end{bmatrix} \begin{bmatrix} cos(\phi_m) & sin(\phi_m) & 0 \\ -sin(\phi_m) & cos(\phi_m) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which multiplies out to:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} cos(\phi_m)cos(\theta_m) & sin(\phi_m)cos(\theta_m) & sin(\theta_m) \\ -sin(\phi_m) & cos(\phi_m) & 0 \\ -cos(\phi_m)sin(\theta_m) & -sin(\phi_m)sin(\theta_m) & cos(\theta_m) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

One can check that this has all the right properties by trying out "special" values of ϕ_m and θ_m, such as 0 and $\pi/2$, combinations of which should rotate different axes over each other, turning, say, a point such as (1,0,0) to (0,1,0) etc., and the matrix therefore should resolve down to various patterns of 0s and 1s. We will, however, leave this after coding the spreadsheet, when it will be part of the testing.

Summary

In summary, therefore:

- We have worked out (probably with Excel) the 3D coordinates (x_n, y_n, z_n) and (x_s, y_s, z_s) from geographical lat. and long. for the two observing stations.
- We apply the 3D rotation matrix derived above (e.g. using Excel) to each of these vectors to get (x'_n, y'_n, z'_n) and (x'_s, y'_s, z'_s) which puts the new x' axis along the Moon/Earth vector, and the new y',z' plane is perpendicular to this line.
- Now ignore $x'_n$and $x'_s$for the moment, and derive the parallax baseline as $\sqrt{(z'_n - z'_s)^2 + (y'_n - y'_s)^2}$ which is a line perpendicular to the Earth/Moon vector.

- Use the measured angular displacement of the Moon to get its distance from the baseline and elementary trig.
- Add the average $(x'_n + x'_s)/2$, to the distance estimated above to get the Moon/Earth, centre-to-centre distance, which is the value quoted in standard references.

In fact, this is still not a *completely* perfect formula, because there are some second-order trigonometry corrections due to not having a perfect isoceles triangle between the two observing stations and the Moon (strincty, we ought to work out all the individual angles in the triangle). However, given that we have a very long thin triangle with two angles with the baseline quite close to 90^o, these corrections will be *much* smaller than the experimental errors, and can safely be ignored.

Testing the Spreadsheet

The above formula have been encoded into an Excel spreadsheet. As with all software it should be tested before being used. If the people who intend to use the software do some of the testing, they can be sure they really understand what the program is supposed to be doing. I have worked as an expert in this area for 40 years and I *always* recommend it as a result of long experience of programs that have errors in them, or programs that may be correct but are doing something a bit different to what I had assumed they are doing. I know I make lots of mistakes, so check, check, check!

One way of doing this is to enter data that are particularly easy to interpret and hand-calculate as a check. So, for example, we might set the Earth's radius to 6000km as an approximation (and an easy number to handle) - but remember to reset it back to 6371km before the real calculation. Then we set the locations of the base stations to locations such as North and South poles, then one the equator at 0, 90, 180, 270 degrees longitude and so on. Similarly, we set the Moon Alt/Azimuth to (0,0), (90,0), (0,90) and so on, to set up simple geometries. All of these produce simple geometries that are easier to visualise (but not that easy!).