Space Elevator Physics

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1 Turning Physics into Equations

I like what I see so far, in terms of getting down to the quantitative aspects of designing a space elevator - but we need to be just a bit more sophisticated with the maths.

In order to understand space elevator physics (that is a continuous cable from a geostationary anchor to Earth) it makes sense to think about small elements of the cable in isolation.

As you have correctly worked out a small test mass moving in a circular track around the Earth of radius r experiences a gravitational force in towards the centre of the Earth of:

$$F_g = \frac{GMm}{r^2} \tag{1}$$

where G is the gravitational constant, M is the mass of the Earth, m is the mass of the cable element. Note that the gravitational force decreases as r increases. Our test mass is, however, undergoing an inward acceleration (a centripetal acceleration due to its circular motion) which would imply a centripetal force of:

$$F_c = mr\omega^2 \tag{2}$$

were ω is the angular velocity $= 2\pi/t$ where t is the period.

We are going to build our space elevator cable out of a series of test masses each rotating with the same angular velocity, that which gives a period of 23 hours 56 minutes. (Note that the Earth rotates on its axis with respect to the fixes stars in slightly less than 24 hours.)

All of our cable elements are moving with the same angular velocity, and we can now look at the imbalance between centripetal force and gravitational force at each radius. This imbalance must be compensated by cable tension to avoid the cable elements moving in or out from the Earth. Clearly, at the bottom of the cable, near the Earth's surface, the cable tension is likely to be very low (and could be zero if there is no anchor mass at the bottom of the cable) since there is no cable mass to support against gravity. Up near the geostationary point it is likely to be very large since a large part of the weight of the cable has to be supported (that part with is not balanced by centripetal forces).

Obviously, near the Earth, gravitational force will be larger than centripetal force, because gravitation varies as $1/r^2$ while centripetal force varies as r. As we go further away centripetal force gets continuously larger and gravitation smaller. As you have pointed out a free test mass can be in a circular orbit where the radius is given by

$$F_c = F_g \tag{3}$$

or

$$mr\omega^2 = \frac{GMm}{r^2} \tag{4}$$

which implies:

$$r = \sqrt[3]{\frac{GM}{\omega^2}} \tag{5}$$

If we set ω to the rotation rate of the Earth we get the geostationary radius.

Now let us think about the net imbalance of force on an element of our cable of length dr. We are going to assume that the cable has a uniform mass per unit length of ρ so the mass of this element is ρdr.

At any radius *except* the geostationary radius there will be an imbalance of forces on this test mass of

$$\frac{GM\rho dr}{r^2} - r\omega^2\rho dr \tag{6}$$

We know that the cable tension has to vary along the cable according to some rule, $T(r)$, increasing as we go from the Earth to the geostationary point, so the imbalance in tension across the cable elements is just

$$T(r+dr) - T(r) = T(r) + \frac{dT}{dr}dr - T(r) = \frac{dT}{dr}dr \tag{7}$$

This net force on the cable element due to the change in tension must then exactly cancel the net imbalance between gravitation and centripetal force.

$$\frac{dT}{dr}dr = \frac{GM\rho dr}{r^2} - r\omega^2\rho dr. \tag{8}$$

or

$$\frac{dT}{dr} = \frac{GM\rho}{r^2} - r\omega^2\rho. \tag{9}$$

Notice the signs: we know these are right because if ω were reduced to zero we only have to take account of gravitation, and tension must increase as we make the cable longer. As we expect the rate of increase in tension is zero at the geostationary radius (where the right hand size reduces to zero).

To work out the explicit formula for tension, just integrate:

$$T(r) = C - \rho \left[\frac{GM}{r} + \frac{r^2\omega^2}{2} \right] \tag{10}$$

where C is the constant of integration.

We can find the value of this constant by setting tension to zero at the Earth's surface (or to any suitable larger value for fixing the cable). Hence:

$$T(r) = \rho \left[\frac{GM}{R} + \frac{R^2\omega^2}{2} \right] - \rho \left[\frac{GM}{r} + \frac{r^2\omega^2}{2} \right] \tag{11}$$

As we might expect the tension in the cable is always directly proportional to the mass per unit length. (By the way: check all my working in detail - I am not infallible and I do sometimes drop signs and constants when I am not being really careful. I am just writing this out as it comes to me and not going back to self-check. Watch out!)

It is interesting to plot this equation. (Do it! It will look good in your report.) The tension clearly starts at zero when $r = R$ then increases as we go out to the geostationary radius, but if we continue beyond this radius it will decrease again (because dT/dr has gone negative beyond the geostationary point - see equation **??**. At some point it would reach zero again, with the outer part of the cable balancing the inner part. Obviously at any point we could cut the outer part of the cable and replace it with a large mass for which the imbalance of centripetal and gravitational force exerts the necessary force on the cable to create the tension required at that point. The larger the counter weight, the closer it could be to the geostationary orbit. Try some examples.

You can now work out the force per unit area (stress) produced by the cable tension by assuming some reasonable material density (say of steel or carbon). You will, of course, find that doubling the surface area of the cable throughout just doubles all the forces so the variation of stress along the cable is only dependent on the material density. Now look up the strength of different materials. These are reported as the maximum stress that can be sustained without stretching or fracture. Compare the strengths for a number of different metals and different forms of carbon (e.g. carbon fibre, carbon nano tubes, graphene and so on).

It should become obvious that we need a low density high strength material.

An obvious problem with making an elevator with a cable of uniform cross section (and uniform mass per unit length) is that the as the tension increases along the cable as we move outwards from the Earth the force per unit area in the cable (the stress) increases. The cable will fail when the stress exceeds the material strength so the cable will fail at the point of maximum stress up near the geostationary point. It is obvious, therefore, that some of the strength on the cable lower down is not necessary: it is extra weight at places where there is no need for extra strength. We can do better by increasing the surface cross section of the cable as the tension increases, keeping the stress uniform.

This is a somewhat more difficult mathematical problem than the uniform cable case. (Because the mass per unit length now has to be proportional to T(r), and therefore appears on the RHS as well as the LHS, the differential equation is not *quite* so easy to integrate. It is, however, only *slightly* more difficult, and within the yr 12 maths syllabus. (If you have not encountered this integral yet you will, so you could look it up in your maths books.) If all else fails there are worked examples for the elevator on the Web. Try it yourself before cheating!

You will still find that the ratio of material strength to material density is the key factor. So which elements could we conceivably make our cable from?

At university, those of you who go on to do physics or material science will learn about the fundamental limitations on material strength, which obviously comes down to the strength of the bonds between planes of atoms. In fact, most materials have observed maximum strengths rather lower than the theoretical maximum, because of atomic level defects in the microscopic structures. Look this up (e.g. Google 'dislocations'). You can also look up 'crack propagation' which prevents glass (which is rather strong) from being a useful engineering material. It is brittle: when you put it under stress, beyond a certain point small cracks in the surface are able to grow into the body of the material at an extremely rapid rate and it shatters. Strength is therefore not the only important engineering criteria. We also like to talk about 'toughness' - the ability to stretch a little before fracturing - as a useful property of desirable engineering materials. Cracks may open up a bit, but do not grow rapidly. E.g. steel is a good material because it is fairly strong *and* tough. (It is also cheap.) However, you might also think about composite materials like glass fibre which enable the strength of glass to be exploited.

We hope that we can create super-strong materials by pushing closer to the theoretical maximum strengths of materials, while also offering toughness (rather than brittleness). Can we do it? Who knows? We can, however, wonder what would be possible if we could do it. (E.g. space elevators? Does the theoretical maximum material strength permit an elevator?)

2 Bye the Way

I notice that you are putting equations into your Word documents by writing them out by hand and then scanning. **This is entirely acceptable for this type of report** and certainly better than not including the equations. It may well be the most efficient way to handle equations for this project, considered in isolation.

In the longer term, however, those of you going on to do physics or engineering or maths etc. are going to find that you need to get on terms with a better method of formatting equations for documents. If you are not going to do a huge amount of such formatting, the Microsoft equation editor is now sort of OK while not being brilliant. (It used to be dire in earlier versions of Word!) While being slightly tedious to use, with a little practice it does work out a good deal faster than the hand-writing+scan-to-image method. It is worth going up the learning curve if you will need the technique in the future (e.g. in an engineering degree). I am not familiar with Open Office equation editors but they appear to be not very dissimilar to Word. While I like Open Office in general, I always do anything with equations in another way, and most people who like working with Open Source software tend to do the same.

Anyone thinking of doing a maths degree will soon get round to tearing out their hair if they try to stick with Word's equation editor. Academic mathematicians and theoretical physicists almost all use an Open Source documentation system called LaTeX. (I am using it for this document.) I recently noted that the Engineering Department at Cambridge recommends LaTeXto its students (but Cambridge Engineerings pulls no punches on its theoretical aspects - other courses focus more strongly on the practical side). If you climb the learning curve (which is not insignificant, but not at all monumental) it makes typing maths as fast as (or faster than) writing equations with pencil and paper. Furthermore, most computer algebra systems will write out LaTeXif you ask nicely, making it is easy for technical professionals to incorporate really complicated maths in documents without errors.

There are, in fact, a number of other reasons why many people prefer

LaTeXto Word when they need to produce highly professional looking documents, even if they do not need equations. This is explained very well in a number of 'LaTeX Advocacy' documents on the Web.

LaTeXis free software and come built in with most Linux systems, and good packages, such as MikTeX are available for Windows.

3 Cables of Varying Cross Section

This net force on the cable element due to the change in tension must still exactly cancel the net imbalance between gravitation and centripetal force, but this time we allow the mass/unit length to vary

$$\frac{dT}{dr} = \frac{GM\rho(r)}{r^2} - r\omega^2\rho(r). \tag{12}$$

Since the mass/unit length is proportional to the cable cross section, and since we want the stress in the cable to be constant along it length we can immediately deduce that the cross section, A(r), must be made proportional to the tension, T, so the mass/unit length must also be proportional to T, that is:

$$\rho(r) = \sigma A(r) = kT(r) \tag{13}$$

where σ is the density of the cable material. If we assume a maximum sustainable stress, S_{max} is the material, then since stress is defines as $S = T/A$ the maximum tension is just $T = S_{max}A$ so

$$k = \frac{\sigma}{S_{max}}. \tag{14}$$

Hence,

$$\frac{dT}{dr} = \frac{GMkT(r)}{r^2} - r\omega^2 kT(r). \tag{15}$$

$$\frac{dT}{T(r)} = k\left[\frac{GM}{r^2} - r\omega^2\right]dr. \tag{16}$$

To work out the explicit formula for tension, just integrate:

$$log(T(r)) = C - k\left[\frac{GM}{r} + \frac{r^2\omega^2}{2}\right] \tag{17}$$

where C is the constant of integration. Which we can write as:

$$T(r) = e^{(C - k\left[\frac{GM}{r} + \frac{r^2\omega^2}{2}\right])} \tag{18}$$

or

$$T(r) = C'e^{-k\left[\frac{GM}{r}+\frac{r^2\omega^2}{2}\right]} \tag{19}$$

At the Earth's surface, we can no longer assume a zero tension, but need to put in some none-zero value, say T_s and we get:

$$\frac{T(r)}{T_s} = e^{k(\left[\frac{GM}{R}+\frac{R^2\omega^2}{2}\right]-\left[\frac{GM}{r}+\frac{r^2\omega^2}{2}\right])} \tag{20}$$

Finally:

$$\frac{T(r)}{T_s} = e^{\frac{\sigma}{S_{max}}(\left[\frac{GM}{R}+\frac{R^2\omega^2}{2}\right]-\left[\frac{GM}{r}+\frac{r^2\omega^2}{2}\right])} \tag{21}$$

As we expect, the critical factor here is the ratio of density to maximum sustainable stress. Note that the term $\left[\frac{GM}{r}+\frac{r^2\omega^2}{2}\right]$ starts large near the Earth's surface and then decreases as r increases, so the tension $T(r)$ and hence the cable cross-section $A = T(r)/S_{max}$ increases rapidly up to geostationary radius.

The limiting factor, now, in cable construction is that total mass of the cable, which increases very rapidly with material density (or reducing maximum sustainable stress). There is no easy integral for the cable mass (because of the 1/r term in the exponential). One has to do it by numerical approximation.